A Note on Pair-Formation Functions

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Abstract

In this paper we take a closer look at two harmonic mean functions [11,14] and two minimum functions (moving dominance function [13] and group-specific minimum function), in two-sex multi-group populations. Comparisons between these functions are focused on proportionate mixing. We show that under some special conditions, the two harmonic mean funtions are identical; and under the mixing framework of Castillo-Chavez and Busenberg [12], the two minimum functions are also identical (for both proportionate and nonproportionate mixing). Simulations of a simple demographic model with the four functions are also performed to confirm the above mentioned identity and to illustrate the behavior of these functions.

Key words: pair formation, marriage function, harmonic mean, minimum function, two-sex population.

1. Introduction

The fast expanding research on sexually transmitted diseases has increased the amount of attention on the role that pair-formation plays in the study of the demographic, ecological and epidemiological processes [1–3]. Some pioneering demographers [4–8] developed the basis on which researchers have constructed pair-formation models. These pioneers were interested in developing (nonlinear) pair-formation functions that exhibited exponential growth and hence used homogeneous functions to model the rates of pair-formation — commonly referred to as marriage functions. Most pair-formation models have been developed to study the dynamics of heterosexual populations that only include one single group of males and females. Two "typical" marriage functions, the minimum function (MF) and the harmonic mean function (HMF), have been applied to many areas, particularly in the dynamics of HIV [1,9,10].

Heterogeneity, in a heterosexually-mixing population, is usually introduced by dividing the population of interest into subgroups (within each sex) based on attributes of interest to the modelers or the scientists (e.g., age, education, etc.). The formulation of appropriate marriage functions under heterogeneity is no longer straightforward. The question of who mixes with whom on a naturally frequency dependent environment has multiple solutions. Several recent articles present various approaches for modeling heterogeneous mixing in two-sex populations [11–15]. In this brief note we focus on describing potential generalizations using the two "typical" marriage functions: MF and HMF. The behavior of these functions are illustrated by simulations of a simple demographic model.

This paper is organized as follows: section 2 gives a brief review of HMF and MF in a two-sex single-group population; section 3 introduces heterogeneity and generalizes HMF and MF in a multi-group population; section 4 describes results of simulations of a simple demographic model; and section 5 collects our thoughts on marriage functions and discusses future research.

2. Brief review of single-group harmonic mean and minimum functions

The earlier work of Kendall [4], Keyfitz [5], Fredrickson [6], MacFarland [7] and Pollard [8] had suggested various functional forms for the rate of pair-formation or marriage function φ, which is a function of the population sizes of single males M and single females F. They extracted a set of basic properties that must be satisfied by the marriage function:

$$\begin{aligned} \varphi(M,F) &\geq 0, \\ \varphi(M+u,F+v) &\geq \varphi(M,F) \text{ for } u,v\geq 0, \\ \varphi(\lambda M,\lambda F) &= \lambda\varphi(M,F) \text{ for } \lambda\geq 0, \\ \varphi(M,0) &= \varphi(0,F)=0. \end{aligned}$$

In 1988 Hadeler, Waldstätter and Wörz-Busekros [16] analyzed their generalized version of the Kendall-Keyfitz pair-formation model. Their analysis was further extended in Waldstätter [10,11]. The Hadeler/Waldstätter/Wörz-Busekros model provides the simplest two-sex demographic model with marriage functions belonging to the class of generalized means. The model is generally nonlinear; however, it is homogeneous of order one and therefore it supports exponential solutions and offers a natural generalization of the Malthus model to two-sex populations. A variety of applications of pair-formation models have been carried out recently [1,9]. Among them the most common selections for $\varphi(M,F)$ are HMF and MF. Generalizations to populations with multiple groups of males and females have also been carried out [10,11,13,14,17], which will be discussed in the next section.

3. Multi-group harmonic mean and minimum functions

We devide the population under consideration into L groups of males and N groups of females. Let M_i and F_j denote respectively the sizes of group i males and group j females, and $\varphi_{ij}(\mathbf{M},\mathbf{F})$ the rate of pair-formation between males of group i and females of group j, where $i=1,\ldots,L$, $j=1,\ldots,N$, $\mathbf{M}=\{M_1,\ldots,M_L\}'$ and $\mathbf{F}=\{F_1,\ldots,F_N\}'$. The folowing two subsections present and compare two specific formulations of the generalized HMF and MF, respectively.

3.1. Generalized harmonic mean function

Using the encounter-mating model of Gimelfarb [18], Waldstätter [11 (chapter 3)] generalizes the HMF to heterogeneously mixing two-sex populations. His approach assumes that every female of group j and every male of group i has a *fixed* average number of contacts per unit time, $\bar{b}_j$ and $\bar{c}_i$, respectively, and the rate of pair-formation between males of group i and females of group j is given by

$$\varphi_{ij}(\mathbf{M},\mathbf{F}) = \rho_{ij}(\mathbf{M},\mathbf{F})\, C^f_{ji}(\mathbf{M},\mathbf{F})\, F_j = \rho_{ij}(\mathbf{M},\mathbf{F})\, C^m_{ij}(\mathbf{M},\mathbf{F})\, M_i \tag{1}$$

with the constraint $C^f_{ji}F_j = C^m_{ij}M_i$, where C^f_{ji} denotes the number of contacts one female of group j has with males of group i, C^m_{ij} denotes the number of contacts one male of group i has with females of group j, and ρ_{ij} denotes the probability that a female of group j will "mate" (form a pair) with a male of group i given that the two individuals had met (encounter each other). The group numbers of contacts are further formulated as $C^f_{ji} = \bar{b}_j\pi^f_{ji}$ and $C^m_{ij} = \bar{c}_i\pi^m_{ij}$, where π^f_{ji} and π^m_{ij} denote the fraction of contacts for males of group i and females of group j, respectively. In general, ρ_{ij}, C^f_{ji}, C^m_{ij}, π^f_{ji} and π^m_{ij} are frequency dependent.

Following the work of Levin and Segel [19] and Waldstätter [10], one sees that function (1) is in fact a generalization of theirs where ρ_{ij} are constant and $\bar{b}_j = \bar{c}_i = k$ for all j and i. Thus, the π's can be alternatively expressed as

$$\pi^f_{ji} = \frac{2\bar{c}_i M_i\tilde{p}_{ij}\tilde{q}_{ji}}{\bar{c}_i M_i\tilde{p}_{ij} + \bar{b}_j F_j\tilde{q}_{ji}} \quad \text{and} \quad \pi^m_{ij} = \frac{2\bar{b}_j F_j\tilde{p}_{ij}\tilde{q}_{ji}}{\bar{c}_i M_i\tilde{p}_{ij} + \bar{b}_j F_j\tilde{q}_{ji}}, \tag{2}$$

where $\tilde{q}_{ji}$ denotes the proportion that females of group j will *encounter* males of group i, $\tilde{p}_{ij}$ denotes the proportion that males of group i will *encounter* females of group j, and both $\tilde{q}_{ji}$ and $\tilde{p}_{ij}$ are frequency dependent.

Proportionate mixing is defined as $\pi^f_{ji} = \bar{c}_i M_i/K \equiv \pi^f_i$ for all j (or $\pi^m_{ij} = \bar{b}_j F_j/K \equiv \pi^m_j$ for all i), where $K = (\sum_{h=1}^N \bar{b}_h F_h + \sum_{k=1}^L \bar{c}_k M_k)/2$ is the total number of contacts by all females and males. This is equivalent to setting

$$\tilde{q}_{ji} = \frac{\bar{c}_i M_i}{\sum_{k=1}^L \bar{c}_k M_k} \quad \text{and} \quad \tilde{p}_{ij} = \frac{\bar{b}_j F_j}{\sum_{h=1}^N \bar{b}_h F_h} \tag{3}$$

in expression (2). Note that $\sum_{i=1}^L \pi^f_i \neq 1$ and $\sum_{j=1}^N \pi^m_j \neq 1$, but $(\sum_{i=1}^L \pi^f_i + \sum_{j=1}^N \pi^m_j)/2 = 1$. The definition of π^f_i (or π^m_j) seems to imply that contacts are modeled as random encounters between males and females; in other words, it is not possible to distinguish a priori whether or not the encounter (chance to meet) is going to be with a male or a female. The corresponding rate of pair-formation under proportionate mixing is then given by

$$\varphi_{ij}(\mathbf{M},\mathbf{F}) = \rho_{ij}(\mathbf{M},\mathbf{F})\, \bar{c}_i\, M_i\, \bar{b}_j\, F_j/K. \tag{4}$$

A different framework of mixing functions has been proposed by Castillo-Chavez and collaborators. Castillo-Chavez and Busenberg [12] present a two-sex mixing framework (CB framework) with the following mixing axioms:

(A1) $0 \leq p_{ji} \leq 1$ and $0 \leq q_{ji} \leq 1$ for all i, j;

(A2) $\sum_{j=1}^N p_{ij} = 1$ for all i, $\sum_{i=1}^L q_{ji} = 1$ for all j;

(A3) $c_i\, M_i\, p_{ij} = b_j\, F_j\, q_{ji}$ for all i, j;

(A4) if $c_i\, b_j\, M_i\, F_j = 0$ for some i or for some j, then $p_{ij} \equiv q_{ji} \equiv 0$ by definition;

where c_i and b_j are per-capita pair-formation rates of group i males and group j females respectively, p_{ij} is the probability that a *partnership* formed by a male of group i is with a female of group j (given that a partnership was formed), q_{ji} is the probability that a *partnership* formed by a group j female is with a male of group i (given that a partnership was formed). Castillo-Chavez and Busenberg [12] further formulate the general solution to (p_{ij}, q_{ji}) as multiplicative perturbation of the proportionate mixing $(\bar{p}_j, \bar{q}_i)$:

$$p_{ij} = \bar{p}_j\, d_{ij} \quad \text{and} \quad q_{ji} = \bar{q}_i\, d_{ij}, \tag{5}$$

where

$$\bar{p}_j = \frac{b_j F_j}{\sum_{h=1}^{N} b_h F_h} \quad \text{and} \quad \bar{q}_i = \frac{c_i M_i}{\sum_{k=1}^{L} c_k M_k}, \tag{6}$$

and d_{ij} is the (frequency dependent) multiplicative perturbation for (i, j) pairs [13,15].

Blythe et al. [20] and Hsu Schmitz [13] point out that within this framework c_i and b_j are generally nonconstant. Castillo-Chavez et al. [14] suggest the following properties for frequency dependent per-capita pair-formation rates:

$$\begin{aligned}
\frac{\partial c_i M_i}{\partial M_i} \geq 0 \quad &\text{and} \quad \frac{\partial c_i}{\partial M_i} \leq 0 \text{ for all } i, \\
\frac{\partial b_j F_j}{\partial F_j} \geq 0 \quad &\text{and} \quad \frac{\partial b_j}{\partial F_j} \leq 0 \text{ for all } j, \\
\frac{\partial c_i}{\partial F_j} \geq 0 \quad &\text{and} \quad \frac{\partial b_j}{\partial M_i} \geq 0 \text{ for all } i, j.
\end{aligned}$$

One of their examples of frequency dependent per-capita pair-formation rates is the following HMF:

$$\begin{aligned}
c_i(\mathrm{M}, \mathrm{F}) &= \frac{\alpha_i \sum_{h=1}^{N} \beta_h F_h}{\sum_{k=1}^{L} \alpha_k M_k + \sum_{h=1}^{N} \beta_h F_h}, \\
b_j(\mathrm{M}, \mathrm{F}) &= \frac{\beta_j \sum_{k=1}^{L} \alpha_k M_k}{\sum_{k=1}^{L} \alpha_k M_k + \sum_{h=1}^{N} \beta_h F_h},
\end{aligned} \tag{7}$$

where α_i $(i = 1, \ldots, L)$ and β_j $(j = 1, \ldots, N)$ are positive constants. The above definition guarantees $\sum_{i=1}^{L} c_i(\mathrm{M}, \mathrm{F})\, M_i = \sum_{j=1}^{N} b_j(\mathrm{M}, \mathrm{F})\, F_j$. Using Axiom (A3) in CB framework the marriage function can then be built as

$$\varphi_{ij}(\mathrm{M}, \mathrm{F}) = c_i(\mathrm{M}, \mathrm{F})\, M_i\, p_{ij} = b_j(\mathrm{M}, \mathrm{F})\, F_j\, q_{ji}. \tag{8}$$

Under proportionate mixing the correponding mixing function $(\bar{p}_j, \bar{q}_i)$ in expression (6) is reduced to

$$\bar{p}_j = \frac{\beta_j F_j}{\sum_{h=1}^{N} \beta_h F_h} \quad \text{and} \quad \bar{q}_i = \frac{\alpha_i M_i}{\sum_{k=1}^{N} \alpha_k M_k}. \tag{9}$$

Note that expression (9) is similar to (3), but with different interpretation. The marriage function then becomes

$$\varphi_{ij} = \frac{\alpha_i M_i \beta_j F_j}{\sum_{k=1}^{L} \alpha_k M_k + \sum_{h=1}^{N} \beta_h F_h}, \tag{10}$$

which is again similar to (4).

If one lets $\rho_{ij} = \rho$ (a constant) for all i, j in function (4), $\alpha_i = 2\rho\, \bar{c}_i$ and $\beta_j = 2\rho\, \bar{b}_j$ in function (10), then these two functions are identical. One obvious choice for ρ is 1/2, which makes $\alpha_i = \bar{c}_i$ and $\beta_j = \bar{b}_j$. This relation implies that the constants α_i and β_j can be estimated from the average numbers of contacts per unit time, $\bar{c}_i$ and $\bar{b}_j$. There is no attemp to compare these two functions under general situations because of the inherent difference between frameworks .

3.2. Generalized minimum function

Although MF has the advantage that stationary states and eigenvalues can be computed explicitly due to its piecewise linearity, the analysis of MF has been carried out only under the assumption that one sex is always more abundant [1]. It is also not clear how one can apply MF to multiple groups in general situations. Hsu Schmitz [13] offers a way of incorporating heterogeneity by assuming that the sex with the smaller total activity is dominant, that is, the sex with smaller total activity is more likely to get its choice. The dominance is thus not fixed in one sex all the times. More specifically, she introduces the "moving dominance" function (MDF)

$$\varphi_{ij}(\mathrm{M}, \mathrm{F}) = \begin{cases} \bar{c}_i\, M_i\, p_{ij} & \text{for all } i,\, j \text{ if } \theta \le 1, \\ \bar{b}_j\, F_j\, q_{ji} & \text{for all } i,\, j \text{ if } \theta \ge 1, \end{cases} \tag{11}$$

where $\bar{c}_i$ $(i = 1, \ldots, L)$ and $\bar{b}_j$ $(j = 1, \ldots, N)$ are positive constants denoting *potential* pairing activity levels (similar to those in Waldstätter's approach in the previous subsection), p_{ij} and q_{ji} satisfy the mixing axioms in CB framework, $\theta = \sum_{k=1}^{L} \bar{c}_k M_k / \sum_{h=1}^{N} \bar{b}_h F_h$ is the total activity ratio. Note that $\bar{c}_i\, M_i\, p_{ij} = \bar{b}_j\, F_j\, q_{ji}$ when $\theta = 1$. The frequency dependent per-capita pair-formation rates are then defined as

$$\begin{aligned} c_i(\mathrm{M}, \mathrm{F}) &= \begin{cases} 0 & \text{if } M_i = 0, \\ \sum_{k=1}^{L} \varphi_{ik}/M_i & \text{otherwise,} \end{cases} \\ b_j(\mathrm{M}, \mathrm{F}) &= \begin{cases} 0 & \text{if } F_j = 0, \\ \sum_{k=1}^{N} \varphi_{kj}/F_j & \text{otherwise.} \end{cases} \end{aligned} \tag{12}$$

From the point of view of available pairing activity, this is a reasonable marriage function because individuals of one sex can not form more partnerships than those available in the opposite sex. However, in this model it is the *total* activities of the two sexes, not the activities of the individual subgroups, which determine the pair-formation rates.

The improved version of MDF in fact has already been suggested by McFarland [7] as the "mutual agreement marriage model" in age-structured populations, which states that the rate of marriages between males of age i and females of age j should be given by the smallest of the corresponding (i, j) group rates of pair-formation. Following our notation, the "mutual agreement marriage model" is expressed as

$$\varphi_{ij}(\mathbf{M}, \mathbf{F}) = \min(\bar{c}_i\, M_i\, p_{ij},\ \bar{b}_j\, F_j\, q_{ji}). \tag{13}$$

We call this function the "group-specific minimum function" (GSMF) as it adjusts the pair-formation rates within each (i,j) group combination.

For both MDF and GSMF, the corresponding proportionate mixing is given by $\bar{p}_j = \bar{b}_j F_j / \sum_{h=1}^{N} \bar{b}_h F_h$ and $\bar{q}_i = \bar{c}_i M_i / \sum_{k=1}^{L} \bar{c}_k M_k$. Using expression (5) one sees that

$$\begin{aligned}
\varphi_{ij}(\mathbf{M}, \mathbf{F}) &= \min(\bar{c}_i\, M_i\, p_{ij},\ \bar{b}_j\, F_j\, q_{ji}) \\
&= d_{ij} \times \min(\bar{c}_i\, M_i\, \bar{p}_j,\ \bar{b}_j\, F_j\, \bar{q}_i) \\
&= \bar{c}_i\, M_i\, \bar{b}_j\, F_j\, d_{ij} \times \min\Big(\frac{1}{\sum_{h=1}^{N} \bar{b}_h F_h}, \frac{1}{\sum_{k=1}^{L} \bar{c}_k M_k}\Big) \\
&= \begin{cases} \bar{c}_i\, M_i\, (\bar{b}_j\, F_j \,/\, \sum_{h=1}^{N} \bar{b}_h F_h)\, d_{ij} & \text{if } \theta \le 1, \\ \bar{b}_j\, F_j\, (\bar{c}_i\, M_i \,/\, \sum_{k=1}^{L} \bar{c}_k M_k)\, d_{ij} & \text{if } \theta \ge 1, \end{cases} \\
&= \begin{cases} \bar{c}_i\, M_i\, p_{ij} & \text{if } \theta \le 1, \\ \bar{b}_j\, F_j\, q_{ji} & \text{if } \theta \ge 1. \end{cases}
\end{aligned}$$

Therefore, the overall total activities of the two sexes decide which is the dominant rate of pair-formation. In other words, the added structure does not add preference heterogeneity under a minimum marriage function. This declares that MDF and GSMF are identical under the CB framework (for both proportionate and nonproportionate mixing).

4. Simulation study

To better understand the behavior of the marriage functions described in the previous section, they are applied to the following demographic model for a simulation study with two groups in each sex:

$$\begin{aligned}
\dot{M}_i &= \Lambda_i^m - \mu^m M_i - \sum_{k=1}^{N} \varphi_{ik} + (\sigma + \mu^f) \sum_{k=1}^{N} P_{ik} \\
\dot{F}_j &= \Lambda_j^f - \mu^f F_j - \sum_{k=1}^{L} \varphi_{kj} + (\sigma + \mu^m) \sum_{k=1}^{L} P_{kj} \\
\dot{P}_{ij} &= \varphi_{ij} - (\sigma + \mu^m + \mu^f) P_{ij}
\end{aligned}$$

where Λ_i^m and Λ_j^f are recruitment rates of unpaired group i males and group j females, μ^m and μ^f are per-capita removal rates of males and females, σ is pair dissolution rate, P_{ij} are numbers of (i,j) pairs, and $i, j = 1, 2$. To demonstrate the effect of total activity ratio, we use oscillating recruitment rates defined as

$$\Lambda_i^m = \kappa_i^m |\sin(t \times 3.6 \times \pi/180)| \text{ and } \Lambda_j^f = \kappa_j^f |\cos(t \times 3.6 \times \pi/180)|,$$

where t is time (0–100) and $\pi = 3.1415926$. The parameters and initial conditions are listed in the appendix. For ease of comparison, we focus on proportionate mixing. To compare the two generalized HMF we let $\rho = 1/2$ in Waldstätter's HMF (W-HMF), and $\alpha_i = \bar{c}_i$ and $\beta_j = \bar{b}_j$ in Castillo-Chavez et al.'s HMF (C-HMF), as suggested in the last paragraph of subsection 3.1.

The results of our simulations confirm that with the given parameters, W-HMF and C-HMF are identical in both rate of pair-formation and population dynamics. Our results also confirm that under CB framework, MDF and GSMF are really identical, not only in dominance pattern but also in rate of pair-formation and population dynamics. Although differentiability could be a problem with MF, we have not encountered difficulties in our simulations.

For all four combinations of pair, HMF gives lower rates of pair-formation than MF (MDF and GSMF) (see fig. 1), which is not a surprise because MF maximizes number of partnerships among those available. Interestingly we observe that for C-HMF and MF the pairing distributions nearly equal the mixing function at all times, that is,

$$\frac{P_{ij}}{\sum_{k=1}^{2} P_{ik}} \cong \bar{p}_j \text{ for all } i \text{ and } \quad \frac{P_{ij}}{\sum_{k=1}^{2} P_{kj}} \cong \bar{q}_i \text{ for all } j.$$

Although this is also true for W-HMF, the interpretation is difficult because it is not constructed under the CB framework.

5. Conclusions

Frequency dependent effects are fundamental in the formulation of any two-sex mixing model for multi-group populations. This was a problem encountered by Ross [21] on his work with vector-transmitted diseases. Demographers have attempted to find reasonable modeling solutions to the two-sex mixing problem, but no major breakthroughs had taken place after the work of Kendall [4], Keyfitz [5], Fredrickson [6], McFarland [7] and Pollard [8]. The study of the epidemiology of HIV brought renewed interest in the field and the work of Dietz, Hadeler and collaborators brought novel solutions to the two-sex problem. The approach that we have followed [14] does not differ substantially from that of Dietz, Hadeler and collaborators except that it seems more flexible for modeling heterogeneous mixing. Moreover, our application to specific mixing data has shown that our models and data are congruent [15,22].

In this brief note we have taken a closer look at possible ways of generalizing marriage functions to two-sex multi-group populations using two specific examples: harmonic mean and minimum function. Under some special conditions, the two generalized harmonic mean functions are identical for proportionate mixing. Due to inherent difference in model construction, the two functions are in fact not easily comparable under general situations. Under the mixing framework of Castillo-Chavez and Busenberg [12], the two generalized minimum functions are shown to be identical (for both porpotionate and nonporpotionate mixing). However, there is no guarantee that they are also identical under other frameworks.

Although the per-capita pair-formation rates are generally frequency dependent, it is often difficult to estimate their dynamics due to lack of longitudinal data. Their point estimates from a single survey are usually applied in research, but one should be aware of the danger of having unbalanced partnerships between the two sexes. The issue in balancing partnerships with per-capita pair-formation rates as constants or from constant probability distributions has been considered in Garnett and Anderson [23] and Kault [24]. The functions discussed in section 3 can be alternative approaches for this problem as well: one can either replace the constants α_i and β_j in Castillo-Chavez et al.'s harmonic mean function (7) with the point estimates of the per-capita pair-formation rates to generate frequency dependent rates, which are then further applied to marriage function (8) to construct partnerships balanced between sexes; or apply the point estimates to either of the two minimum functions ((11) or (13)) to generate balanced partnerships, then backward construct the frequency dependent per-capita pair-formation rates as in (12).

We have begun to explore the use of specific marriage/mating functions in population dynamic models that include genetics. Our preliminary results using the above approach are quite promising and they will be reported elsewhere in the near future.

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Appendix: Initial conditions and parameter values of the simulations

$$
\begin{aligned}
&M_1(0) = 200,\ M_2(0) = 1000,\ F_1(0) = 300,\ F_2(0) = 1500,\\
&P_{11}(0) = P_{12}(0) = P_{21}(0) = P_{22}(0) = 5,\\
&\bar{c}_1 = 6,\ \bar{c}_2 = 3,\ \bar{b}_1 = 4,\ \bar{b}_2 = 1.7,\\
&\kappa_1^m = 50,\ \kappa_2^m = 100,\ \kappa_1^f = 20,\ \kappa_2^f = 150,\\
&\mu^m = 0.05,\ \mu^f = 0.04,\ \sigma = 4.
\end{aligned}
$$

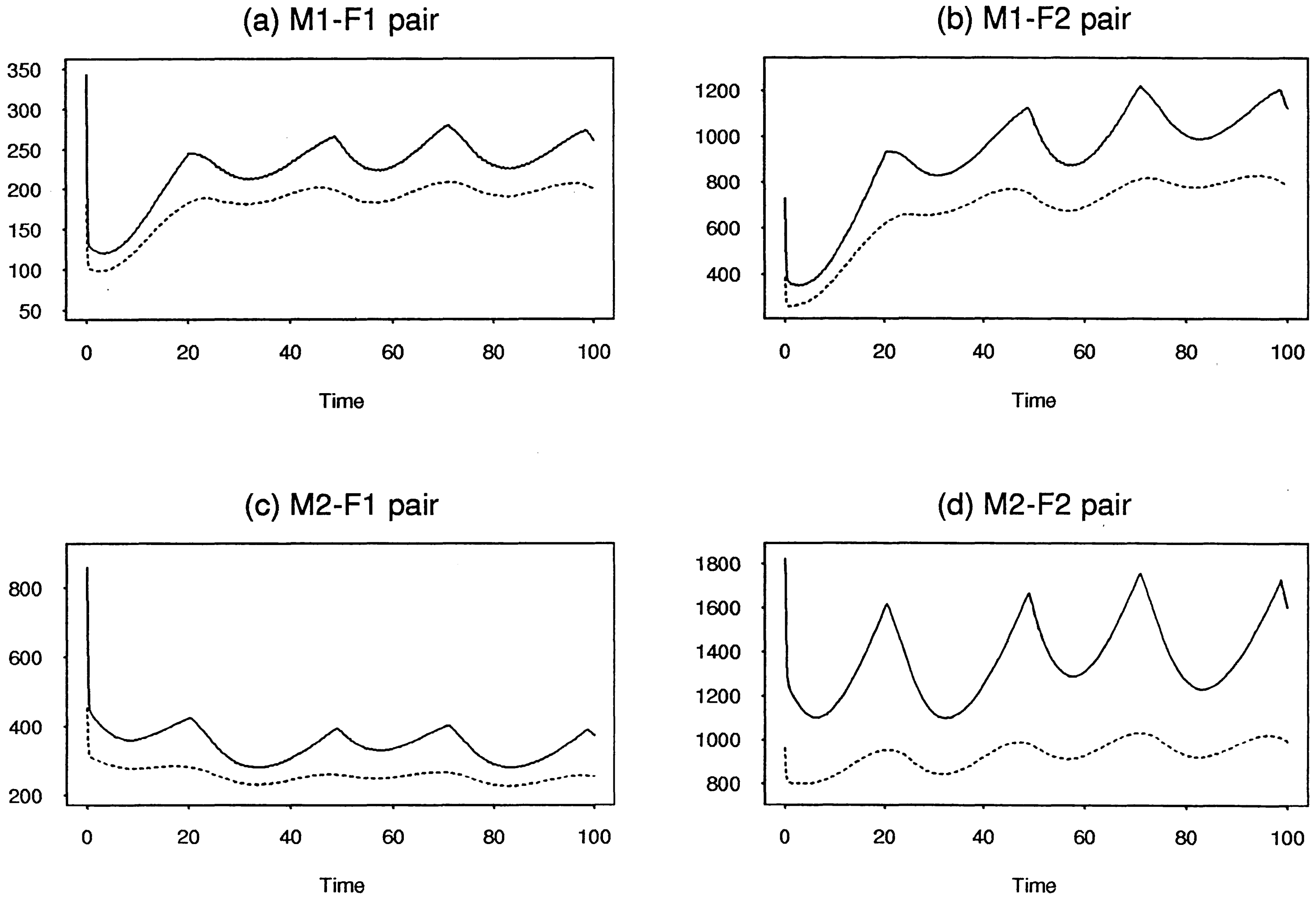

Figure 1. Rate of pair formation (solid line - MF, dotted line - HMF)